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Hamiltonian character of the motion of the zeros of a polynomial whose coefficients oscillate over time

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Abstract. A Hamiltonian is explicitly exhibited, whose equations of motion yield the time evolution of the n zeros, $z_j(t)$, of a polynomial of degree n in z, $P_n(z, t) = z^n + \sum_{m=1}^{n} c_m(t)z^{n-m}$, when its coefficients $c_m(t)$ oscillate, $c_m(t) = c_m^{(+)}\exp(\mathrm{i}\omega_m t) + c_m^{(-)}\exp(-\mathrm{i}\omega_m t)$, or evolve in some other Hamiltonian manner.

1. Introduction

Consider the polynomial of degree n in z,

$$P_n(z,t) = z^n + \sum_{m=1}^{n} c_m(t)z^{n-m} \tag{1.1}$$

and assume that its coefficients $c_m(t)$ oscillate over time,

$$c_m(t) = c_m^{(+)}\exp(\mathrm{i}\omega_m t) + c_m^{(-)}\exp(-\mathrm{i}\omega_m t). \tag{1.2}$$

It is then rather evident [1] that the evolution over time of the zeros $z_j(t)$ of this polynomial,

$$P_n[z_j(t), t] = 0 \qquad j = 1, \ldots, n \tag{1.3}$$

is Hamiltonian, namely that there exists a Hamiltonian function $H(\boldsymbol{z}, \boldsymbol{v})$ such that the motion of the n zeros $z_j(t)$ is given by the Hamiltonian equations

$$\dot{z}_j = \partial H(\boldsymbol{z}, \boldsymbol{v})/\partial v_j \qquad j = 1, \ldots, n \tag{1.4a}$$

$$\dot{v}_j = -\partial H(\boldsymbol{z}, \boldsymbol{v})/\partial z_j \qquad j = 1, \ldots, n \tag{1.4b}$$

(with appropriate initial conditions, see below). The purpose and scope of this paper is to provide (in the following section) an *explicit* representation of the Hamiltonian $H(\boldsymbol{z}, \boldsymbol{v})$. The next section outlines some extensions, exhibits some simple examples (for $n = 2$ and $n = 3$), and discusses briefly the relations of the results of this paper with some recent findings [2]. At the end we have added for completeness an appendix which reviews some standard facts on Hamiltonian systems and symplectic geometry.

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2. The Hamiltonian $H(z, v)$

The time evolution (1.2) of the coefficients $c_m(t)$ is clearly Hamiltonian, corresponding to the equations

$$\dot{c}_m = \partial\tilde{H}(\boldsymbol{c}, \boldsymbol{p})/\partial p_m \qquad m = 1, \ldots, n \tag{2.1a}$$

$$\dot{p}_m = -\partial\tilde{H}(\boldsymbol{c}, \boldsymbol{p})/\partial c_m \qquad m = 1, \ldots, n \tag{2.1b}$$

with

$$\tilde{H}(\boldsymbol{c}, \boldsymbol{p}) = \frac{1}{2}\sum_{m=1}^{n}(p_m^2 + \omega_m^2 c_m^2) \tag{2.2}$$

and to the initial conditions

$$c_m(0) = c_m^{(+)} + c_m^{(-)} \tag{2.3a}$$

$$\dot{c}_m(0) = \mathrm{i}\omega_m(c_m^{(+)} - c_m^{(-)}). \tag{2.3b}$$

The n coefficients c_m are related to the n zeros z_j by the algebraic relations implied by the equation

$$z^n + \sum_{m=1}^{n} c_m z^{n-m} = \prod_{j=1}^{n}(z - z_j). \tag{2.4}$$

Hence, in the framework of the Hamiltonian formalism, the transformation from the n coordinates c_m to the n coordinates z_j is a *point* transformation; there therefore exist n canonical momenta v_j such that the transformation from the $2n$ coordinates and momenta c_m, p_m to the $2n$ coordinates and momenta z_j, v_j is *canonical*, entailing that the time evolution of the quantities $z_j(t)$ and $v_j(t)$ is given by the Hamiltonian equations (1.4) with

$$H(\boldsymbol{z}, \boldsymbol{v}) = \tilde{H}[\boldsymbol{c}(\boldsymbol{z}), \boldsymbol{p}(\boldsymbol{z}, \boldsymbol{v})] \tag{2.5}$$

of course with $\tilde{H}(\boldsymbol{c}, \boldsymbol{p})$ given by (2.2). To obtain $H(\boldsymbol{z}, \boldsymbol{v})$ one must therefore find the expressions of the (old) *canonical* coordinates c_m and momenta p_m in terms of the (new) *canonical* coordinates z_j and momenta v_j.

The expressions of the coefficients c_m of a polynomial in terms of its zeros z_j are well known, being directly implied by (2.4):

$$c_1 = -\sum_{j=1}^{n} z_j \tag{2.6a}$$

$$c_2 = \sum_{j,k=1;\, j\neq k}^{n} z_j z_k \tag{2.6b}$$

$$c_3 = -\sum_{j,k,\ell=1;\, j\neq k, k\neq\ell, \ell\neq j}^{n} z_j z_k z_\ell \tag{2.6c}$$

and so on.

The expressions of the canonical momenta p_m in terms of the coordinates and momenta z_j, v_j is characterized by the requirement that the transformation $(\boldsymbol{c}, \boldsymbol{p}) \to (\boldsymbol{z}, \boldsymbol{v})$ be canonical, namely by the condition

$$\{c_m, p_{m'}\} = \delta_{mm'} \tag{2.7}$$

with the standard definition of the Poisson bracket,

$$\{c_m, p_{m'}\} \equiv \sum_{j=1}^{n} \left\{ \frac{\partial c_m}{\partial z_j} \frac{\partial p_{m'}}{\partial v_j} - \frac{\partial c_m}{\partial v_j} \frac{\partial p_{m'}}{\partial z_j} \right\}. \tag{2.8}$$

This entails, in our case,

$$\sum_{j=1}^{n} \frac{\partial c_m(\boldsymbol{z})}{\partial z_j} \frac{\partial p_{m'}(\boldsymbol{z}, \boldsymbol{v})}{\partial v_j} = \delta_{mm'}. \tag{2.9}$$

It is therefore clear that p_m is linear in $\boldsymbol{v}$,

$$p_m = \sum_{j=1}^{n} C_{mj}(z) v_j + r_m(z) \tag{2.10}$$

where $r_m(z)$ is an arbitrary function of z. In the framework of symplectic geometry, it is appropriate (see the appendix) to choose $r_m(z) = 0$. Clearly (2.10) implies

$$\partial p_m / \partial v_j = C_{mj}(\boldsymbol{z}) \tag{2.11}$$

and since obviously

$$\partial c_m / \partial c_{m'} = \delta_{mm'} = \sum_{j=1}^{n} \frac{\partial c_m(\boldsymbol{z})}{\partial z_j} \frac{\partial z_j(\boldsymbol{c})}{\partial c_{m'}} \tag{2.12}$$

it is, moreover, clear from (2.9), (2.11) and (2.12) that

$$C_{mj}(\boldsymbol{z}) = \partial z_j(\boldsymbol{c}) / \partial c_m. \tag{2.13}$$

To calculate the quantities (2.13), it is convenient to differentiate (2.4) with respect to c_m. This yields

$$z^{n-m} = -\sum_{\ell=1}^{n} (\partial z_\ell / \partial c_m) \prod_{k=1, k\neq\ell}^{n} (z - z_k). \tag{2.14}$$

Hence, by setting $z = z_j$, one immediately obtains the (presumably well known) formula

$$\partial z_j / \partial c_m = -z_j^{n-m} / P_n'(z_j) \tag{2.15}$$

with

$$P_n'(z_j) = \prod_{k=1, k\neq j}^{n} (z_j - z_k). \tag{2.16}$$

The explicit expression of p_m in terms of z_j and v_j therefore reads,

$$p_m = -\sum_{j=1}^{n} v_j z_j^{n-m} / P_n'(z_j). \tag{2.17}$$

Hence, the explicit expression of the Hamiltonian $H(\boldsymbol{z}, \boldsymbol{v})$ reads (from (2.2) and (2.5))

$$H(\boldsymbol{z}, \boldsymbol{v}) = \frac{1}{2} \sum_{j,k=1}^{n} g_{jk}(\boldsymbol{z}) v_j v_k + \frac{1}{2} \sum_{m=1}^{n} \omega_m^2 c_m^2(\boldsymbol{z}) \tag{2.18a}$$

with

$$g_{jk}(\boldsymbol{z}) = \{[1 - (z_j z_k)^n]/(1 - z_j z_k)\}/[P_n'(z_j) P_n'(z_k)] \tag{2.18b}$$

of course with (2.16) and the coefficients $c_m(\boldsymbol{z})$ given by (2.6).

Two final remarks.

• Clearly the Hamiltonian (2.18) (with (2.16) and (2.6)) is completely integrable, and possesses the n integrals of motion in involution

$$H_m(\boldsymbol{z},\boldsymbol{v}) = \tfrac{1}{2}\{[p_m(\boldsymbol{z},\boldsymbol{v})]^2 + \omega_m^2[c_m(\boldsymbol{z})]^2\} \qquad m = 1,\ldots,n \tag{2.19}$$

with $p_m(\boldsymbol{z},\boldsymbol{v})$ and $c_m(\boldsymbol{z})$ given by (2.17) and (2.16) and by (2.6).

• The time evolution of the n zeros $z_j(t)$ of the polynomial (1.1) with (1.2) is given by the specific solution of the Hamiltonian equations of motion (1.4) with (2.18) (and (2.16) and (2.6)), *characterized by the initial conditions (1.3)*, which clearly entail that the n initial values $z_j(0)$ are the n zeros of the polynomial $P_n(z,0)$ (see (1.1); of course with (2.3*a*)), and the n initial values $v_j(0)$ are the solutions of the system of n linear equations

$$\sum_{j=1}^{n} v_j(0)[z_j(0)]^{n-m}/P_n'[z_j(0)] = -\mathrm{i}\omega_m[c_m^{(+)} - c_m^{(-)}] \qquad m = 1,\ldots,n. \tag{2.20}$$

Here $P_n'[z_j(0)]$ is given by (2.16) (at $t = 0$), or equivalently (see (1.1) and (2.4)) by the expression

$$P_n'(z,t) = \partial P_n(z,t)/\partial z = nz^{n-1} + \sum_{m=1}^{n}(n-m)c_m(t)z^{n-m} \tag{2.21}$$

evaluated at $t = 0$ and $z = z_j(0)$. Note that (2.20) corresponds to (2.17) via (2.3*b*), since (2.1*a*) with (2.2) entail

$$p_m = \dot{c}_m. \tag{2.22}$$

An alternative assignment of the initial conditions provides the values of the 'initial velocities' $\dot{z}_j(0)$. They are given by the expressions

$$\dot{z}_j(0) = -\mathrm{i}\left\{\sum_{m=1}^{n}\omega_m[c_m^{(+)} - c_m^{(-)}][z_j(0)]^{n-m}\right\}\Big/ P_n'[z_j(0)] \tag{2.23}$$

which are obtained by time differentiating (2.4) and then setting $t = 0$ and $z = z_j(0)$ (and using (2.3*b*)). For $P_n'[z_j(0)]$ one can use here either expression (2.16) or (2.21).

3. Extensions, examples and discussion

It is clear that the treatment given above is immediately extendable, with obvious modifications, to *any* time evolution of the coefficients $c_m(t)$ (other than (1.2)) which is *Hamiltonian*.

While it should be emphasized that the interest (if any) of the results reported above rests on the possibility of treating the case of arbitrary (positive integer) $n \geqslant 2$, we display below the formulae for the simple cases of $n = 2$ and $n = 3$.

For $n = 2$,

$$H(\boldsymbol{z},\boldsymbol{v}) = \tfrac{1}{2}[(v_1 - v_2)^2 + (z_1v_1 - z_2v_2)^2]/(z_1 - z_2)^2 + \tfrac{1}{2}\omega_1^2(z_1 + z_2)^2 + \tfrac{1}{2}\omega_2^2(z_1z_2)^2 \tag{3.1}$$

$$\dot{z}_1 = [v_1 - v_2 + (z_1v_1 - z_2v_2)z_1]/(z_1 - z_2)^2 \tag{3.2a}$$

$$\dot{z}_2 = -[v_1 - v_2 + (z_1v_1 - z_2v_2)z_2]/(z_1 - z_2)^2 \tag{3.2b}$$

$$\dot{v}_1 = (v_1 - v_2)[v_1 - v_2 + (z_1v_1 - z_2v_2)z_2]/(z_1 - z_2)^3 - \omega_1^2(z_1 + z_2) - \omega_2^2 z_1 z_2^2 \tag{3.3a}$$

$$\dot{v}_2 = (v_1 - v_2)[v_1 - v_2 + (z_1v_1 - z_2v_2)z_1]/(z_1 - z_2)^3 - \omega_1^2(z_1 + z_2) - \omega_2^2 z_1^2 z_2. \tag{3.3b}$$

Note that these equations entail the relations

$$\dot{v}_1 = -[(v_1 - v_2)/(z_1 - z_2)]\dot{z}_2 - \omega_1^2(z_1 + z_2) - \omega_2^2 z_1 z_2^2 \tag{3.4a}$$

$$\dot{v}_2 = -[(v_1 - v_2)/(z_1 - z_2)]\dot{z}_1 - \omega_1^2(z_1 + z_2) - \omega_2^2 z_1^2 z_2 \tag{3.4b}$$

and the second-order equations

$$\ddot{z}_j + \omega_1^2 z_j = [2\dot{z}_j\dot{z}_{j+1} + (\omega_2^2 - 2\omega_1^2)z_j z_{j+1}]/(z_j - z_{j+1}) \qquad j = 1, 2 \bmod(2). \tag{3.5}$$

For $n = 3$,

$$\begin{aligned} H(\boldsymbol{z}, \boldsymbol{v}) = {} & \frac{1}{2}\sum_{j=1}^{3}(1 + z_j^2 + z_j^4)v_j^2/[(z_j - z_{j+1})^2(z_j - z_{j+2})^2] \\ & - \sum_{j=1}^{3}(1 + z_{j+1}z_{j+2} + z_{j+1}^2 z_{j+2}^2)v_{j+1}v_{j+2}/[(z_{j+1} - z_{j+2})^2(z_j - z_{j+1})(z_j - z_{j+2})] \\ & + \tfrac{1}{2}[\omega_1^2(z_1 + z_2 + z_3)^2 + \omega_2^2(z_1z_2 + z_2z_3 + z_3z_1)^2 + \omega_3^2(z_1z_2z_3)^2] \end{aligned} \tag{3.6}$$

$$\begin{aligned} \ddot{z}_j + \omega_1^2 z_j = {} & \sum_{k=1, k\neq j}^{3} [2\dot{z}_j\dot{z}_k + (\omega_2^2 - 2\omega_1^2)z_j z_k]/(z_j - z_k) \\ & + (-3\omega_1^2 + 3\omega_2^2 - \omega_3^2)z_1z_2z_3/[(z_j - z_{j+1})(z_j - z_{j+2})] \qquad j = 1, 2, 3. \end{aligned} \tag{3.7}$$

In the last two equations the index j is defined mod(3).

The connection between the findings reported in this paper and those entailed by the approach of [1] can easily be traced to the fact that the polynomial $P_n(z, t)$, see (1.1) and (1.2), satisfies the linear partial differential equation

$$\left[\partial^2/\partial t^2 + \sum_{r=0}^{n} b_r z^r \partial^r/\partial z^r\right] P_n(z, t) = 0 \tag{3.8}$$

with the $n + 1$ coefficients b_r defined by the linear triangular system

$$\sum_{r=0}^{n-m} b_r (n - m)!/(n - m - r)! = \omega_m^2 \qquad m = n, \ldots, 1 \tag{3.9a}$$

$$\sum_{r=0}^{n} b_r/(n - r)! = 0. \tag{3.9b}$$

It is easily seen that the condition that (3.8) be a second-order partial differential equation (PDE) (namely, that $b_r = 0$ for $r > 2$) entails, via (3.9), the formula

$$\omega_m^2 = -m[b_1 + (2n - m - 1)b_2] \tag{3.10}$$

which is consistent, via (1.2) and up to trivial notational changes, with equation (4.13) of [2]. Note that the condition that (3.8) be a *second-order* PDE is necessary and sufficient to guarantee [1] that the 'Newtonian' second-order equations,

$$\ddot{z}_j = F_j(\boldsymbol{z}, \dot{\boldsymbol{z}}) \tag{3.11}$$

entailed by the Hamiltonian equations (1.4), feature 'forces' F_j that contain only 'one-body' and 'two-body' terms,

$$F_j(\boldsymbol{z}, \dot{\boldsymbol{z}}) = f^{(1)}(z_j, \dot{z}_j) + \sum_{k=1, k\neq j}^{n} f^{(2)}(z_j, z_k, \dot{z}_j, \dot{z}_k). \tag{3.12}$$

Indeed, the diligent reader may easily check that, for $n = 3$, (3.10) entails the constraint

$$\omega_3^2 - 3\omega_2^2 + 3\omega_1^2 = 0 \tag{3.13}$$

which is indeed necessary and sufficient to eliminate the 'three-body' forces from the right-hand side of (3.7).

Up to now we have omitted to specify whether the context of our treatment was real or complex. It is in fact obvious that the most appropriate context to investigate the zeros of a polynomial is complex. Indeed all that has been written above is applicable in a completely complex context, that is assuming not only that the quantities c_m, p_m and z_j, v_j are complex, but that even the 'frequencies' ω_m are complex. In such a case, of course, the coefficients $c_m(t)$ would not merely oscillate over time, but might diverge to infinity or converge to zero as $t \to \pm\infty$. The corresponding behaviour in such cases of the zeros $z_j(t)$ is discussed in appendix A of [2].

The results reported above yield, via such complexification, a large family of *integrable Hamiltonian models*, which are naturally interpretable as describing the motion of n interacting particles *in the plane*, whose evolution is determined by *rotation-invariant* equations of motion of Newtonian type [3]. These models include all the Hamiltonian models introduced in [2], which are characterized by the restriction to feature only one-body and two-body interparticle forces.

We end this paper with a remark that has been added at the request of a referee. Whenever two zeros coincide, the canonical transformation discussed in this paper becomes singular, as evidenced by the vanishing of the denominator on the right-hand side of (2.18*b*). If the motion of the zeros is interpreted as that of particles in the context of a (solvable) many-body problem, this phenomenon has a natural interpretation as a two-body collision, after which the Hamiltonian motion may or may not be continued, depending on the context [1–3].

Appendix

The expression of the canonical momenta p_m in terms of the coordinates and momenta (z_j, v_j) is characterized by the requirement that the transformation $(\boldsymbol{c}, \boldsymbol{p}) \to (\boldsymbol{z}, \boldsymbol{v})$ be canonical, namely by the condition

$$\sum_{m=1}^{n} \mathrm{d}c_m \wedge \mathrm{d}p_m = \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}v_j \tag{A.1}$$

(conservation of the canonical symplectic 2-form). This entails that there exists a function $S(\boldsymbol{c}, \boldsymbol{v})$ such that

$$\sum_{m=1}^{n} p_m\, \mathrm{d}c_m + \sum_{j=1}^{n} z_j\, \mathrm{d}v_j = \mathrm{d}S(\boldsymbol{c}, \boldsymbol{v}) \tag{A.2a}$$

$$\frac{\partial S}{\partial c_m} = p_m \qquad \frac{\partial S}{\partial v_j} = z_j. \tag{A.2b}$$

In our case, $\boldsymbol{z}$ depends only on $\boldsymbol{c}$ and so we obtain

$$S(\boldsymbol{c}, \boldsymbol{v}) = \sum_{j=1}^{n} z_j v_j + S_0(\boldsymbol{c}) \tag{A.3}$$

hence

$$p_m = \sum_{j=1}^{n} \frac{\partial z_j}{\partial c_m} v_j + \frac{\partial S_0}{\partial c_m}. \tag{A.4}$$

One can choose S_0 arbitrarily. In particular, one can set $S_0 = 0$. In this manner, one lifts the *point transformation* $\boldsymbol{c} \to \boldsymbol{z}$ to a *contact transformation* (i.e. $(\boldsymbol{c}, \boldsymbol{p}) \to (\boldsymbol{z}, \boldsymbol{v})$ not only preserves the symplectic 2-form but also the Liouville 1-form, $\sum_m p_m \, \mathrm{d}c_m = \sum_j v_j \, \mathrm{d}z_j$).

Definition. A Hamiltonian system H on $(\mathbb{R}^{2m}, \sum_{j=1}^{m} \mathrm{d}x_j \wedge \mathrm{d}p_j)$ defines a Newtonian dynamics if, given Hamilton's first-order equations

$$\dot{c}_m = \frac{\partial H}{\partial p_m} \tag{A.5a}$$

$$\dot{p}_m = -\frac{\partial H}{\partial c_m} \tag{A.5b}$$

the solution $(c_m(t),\ p_m(t))$ with initial conditions $(c_m(0),\ p_m(0))$ corresponds to the unique solution $(c_m(t),\ \dot{c}_m(t))$ of a system of second-order differential equations

$$\ddot{c}_m = f_m(\boldsymbol{c}, \dot{\boldsymbol{c}}) \tag{A.6}$$

with initial values $(c_m(0),\ \dot{c}_m(0))$, where

$$\dot{c}_m(0) = \frac{\partial H}{\partial p_m}(\boldsymbol{c}(0), \boldsymbol{p}(0)). \tag{A.7}$$

The vector $\boldsymbol{f} = (f_m)$ is then called the force.

Two remarks are now in order.

(i) Different Hamiltonian systems may lead to the same Newtonian dynamics (A.6).

(ii) The simplest Newtonian case is given by the natural Hamiltonian

$$H = \frac{1}{2}\sum_{m=1}^{n} p_m^2 + V(\boldsymbol{c}) \tag{A.8}$$

which yields

$$\dot{c}_m = p_m \qquad \dot{p}_m = -\frac{\partial V}{\partial c_m} \tag{A.9}$$

hence

$$\ddot{c}_m = -\frac{\partial V}{\partial c_m}(\boldsymbol{c}). \tag{A.10}$$

In this case the force does not depend on the velocity.

We conclude with two propositions on Newtonian dynamics and contact transformations.

Proposition 1. Let H be a Hamiltonian system of type

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(\boldsymbol{c}) p_i p_j + V(\boldsymbol{c}) \tag{A.11}$$

with the matrix $\mathsf{G} = (g_{ij})$ invertable. Then H defines a Newtonian dynamics.

Proof. Hamilton's equations yield

$$\dot{\boldsymbol{c}} = \mathsf{G}(\boldsymbol{c})\boldsymbol{p} \tag{A.12a}$$

hence

$$\boldsymbol{p} = \mathsf{G}^{-1}(\boldsymbol{c})\dot{\boldsymbol{c}}. \tag{A.12b}$$

This entails, from (A.12*a*) and (A.5*b*),

$$\ddot{c}_m = \sum_{j,k=1}^{n} \frac{\partial g_{mj}}{\partial c_k} \dot{c}_k (\mathbf{G}^{-1}\boldsymbol{c})_j - \sum_{j=1}^{n} g_{mj}(\boldsymbol{c}) \left[\frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial g_{kl}(\boldsymbol{c})}{\partial c_m} (\mathbf{G}^{-1}(\boldsymbol{c})\dot{\boldsymbol{c}})_k (\mathbf{G}^{-1}(\boldsymbol{c})\dot{\boldsymbol{c}})_l + \frac{\partial V(\boldsymbol{c})}{\partial c_m} \right]. \tag{A.13}$$

□

Proposition 2. A Hamiltonian system of type (A.11) gets transformed into another Hamiltonian system of the same type under a contact transformation.

Proof. Such a transformation is

$$(\boldsymbol{c}, \boldsymbol{p}) \rightarrow (\boldsymbol{z}, \boldsymbol{v}) \qquad \boldsymbol{z} = \boldsymbol{z}(\boldsymbol{c}) \qquad p_m = \sum_{j=1}^{n} \frac{\partial z_j}{\partial c_m} v_j \tag{A.14}$$

where the matrix $\mathbf{T}$, of elements $t_{jm} = \partial z_j / \partial c_m$, is invertable. Then one may write

$$H = \tfrac{1}{2} \langle \mathbf{G}\boldsymbol{p}, \boldsymbol{p} \rangle + V(\boldsymbol{c}) \tag{A.15}$$

$$\boldsymbol{p} = \mathbf{T}\boldsymbol{v} \tag{A.16}$$

$$H = \tfrac{1}{2} \langle {}^t\mathbf{TGT}\boldsymbol{v}, \boldsymbol{v} \rangle + V(\boldsymbol{c}(\boldsymbol{z})). \tag{A.17}$$

The matrices $\mathbf{G}$ and $\mathbf{T}$ are invertable, hence ${}^t\mathbf{TGT}$ is also invertable. □

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